Design without CAD Exchange – A Secure Procedure for Design and Analysis in Distributed Product Development Environments

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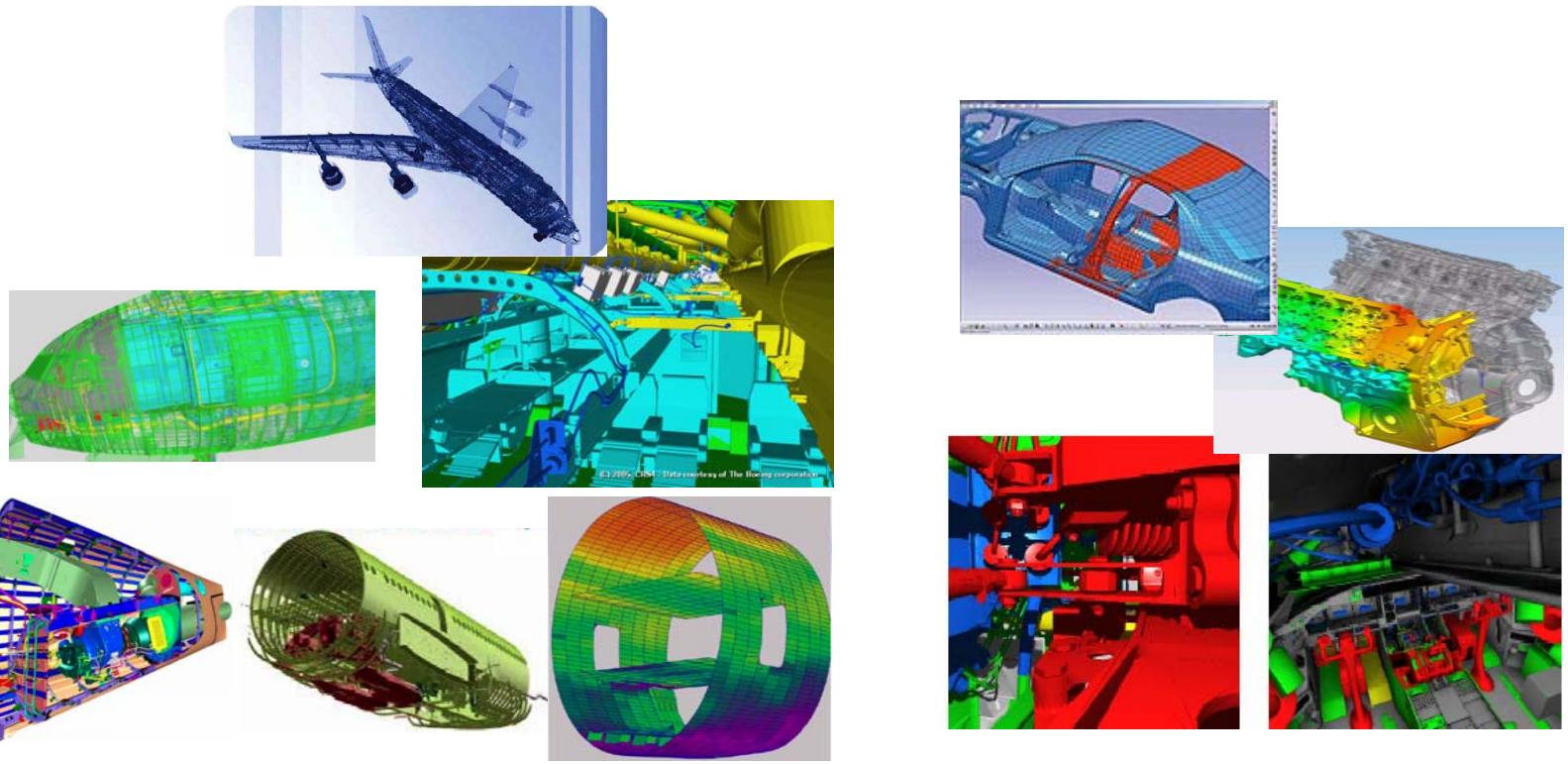
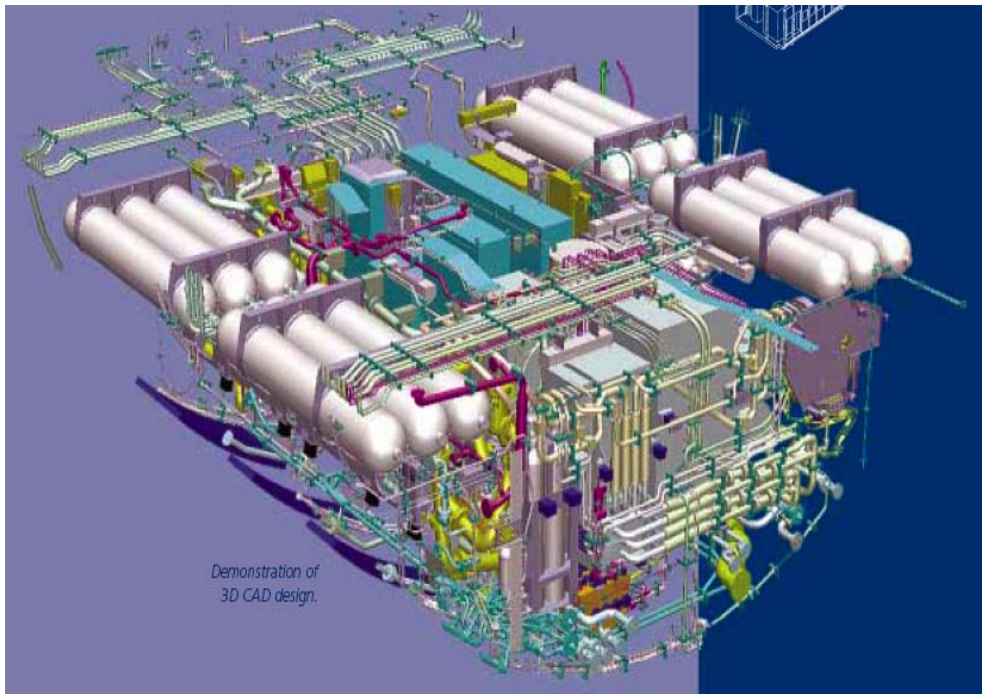
Nature of Distributed Product Development

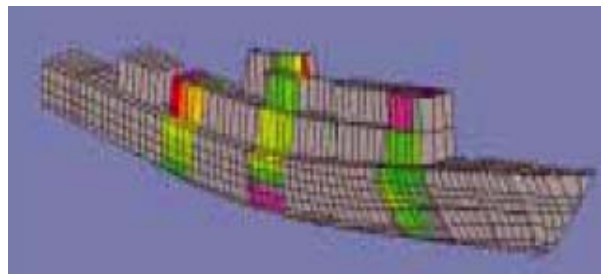
- Systems are complex
- Include multiple interacting subsystems
- Subsystems are independently designed
- Interaction of subsystems determines overall system performance



Example: Ship Design



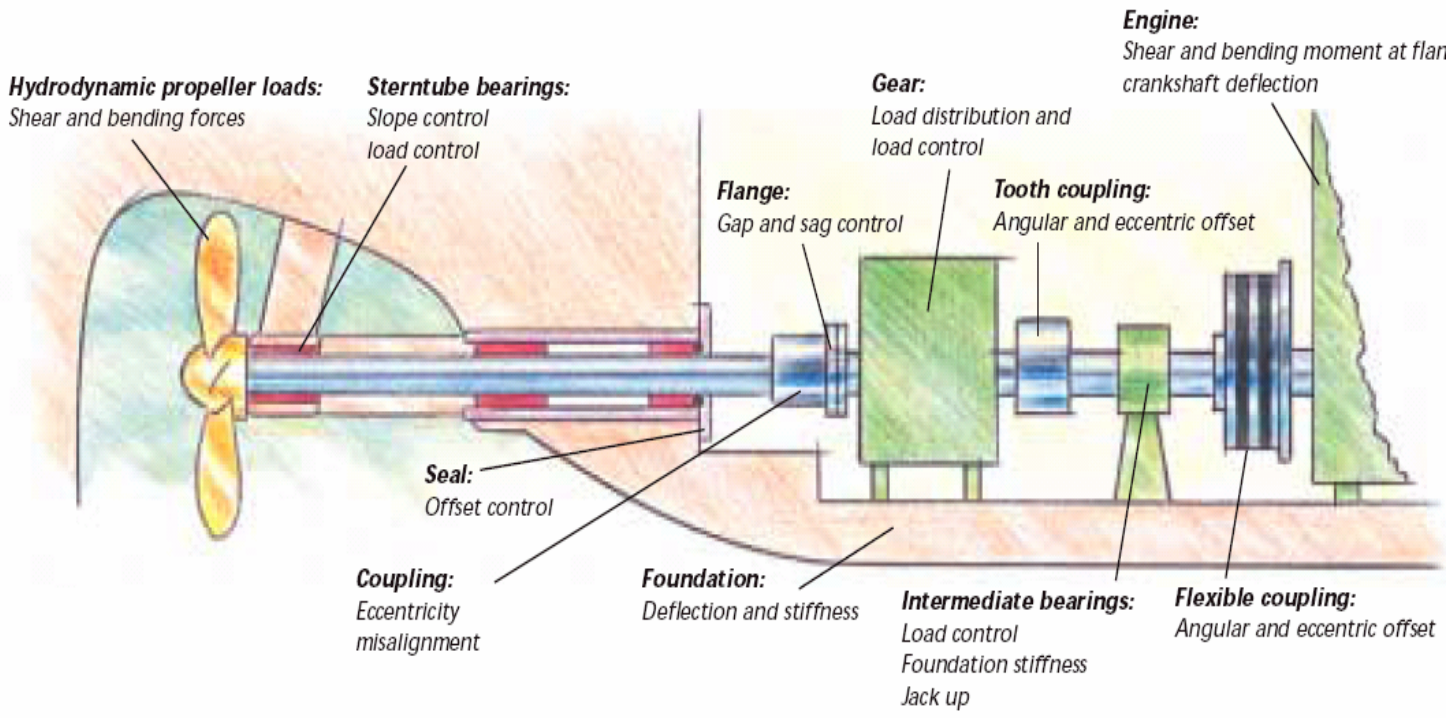
3-D CAD Design



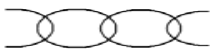
Finite Element Analysis

http://www.asc.com.au/cms_resources/documents/corporatecommunications/shipbuilding_capability.pdf

Propulsion System



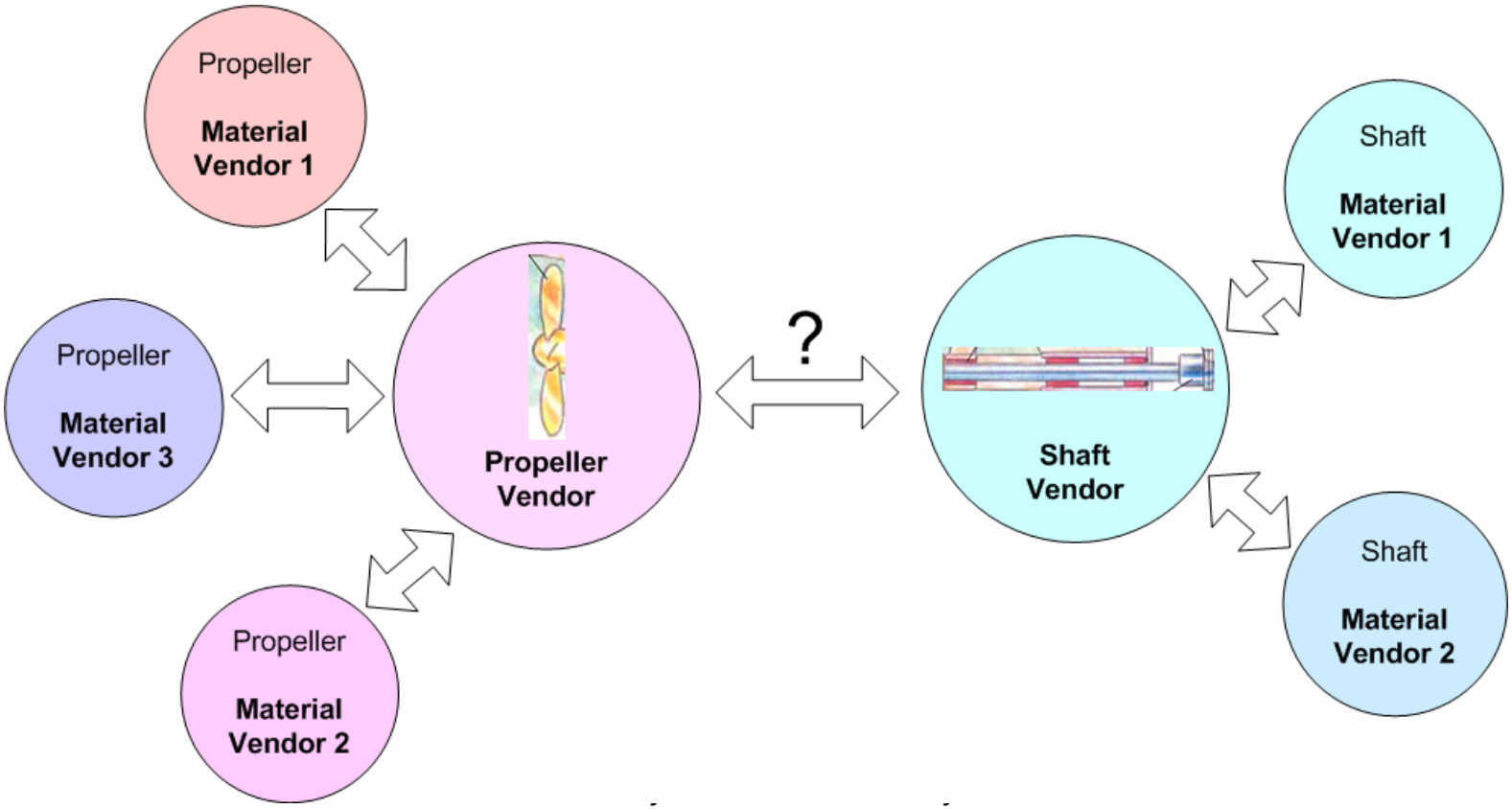
http://www.dnv.com/binaries/mach_SHAFT_tcm4-10151.pdf



Design and
Behavior
Constraint

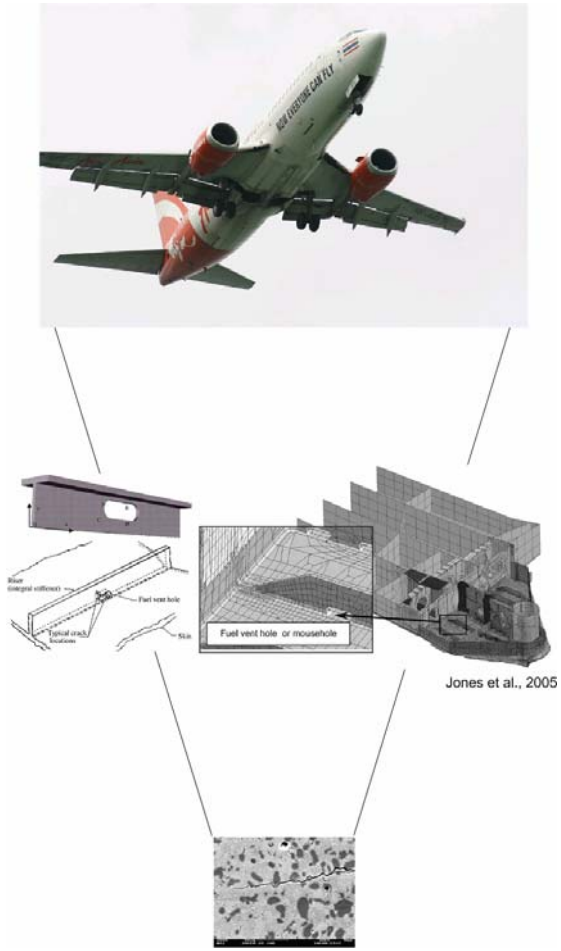
Distributed Design Scenario

- System level behavior: Maximum Stress
- Geometric and Material Properties of Shaft and Propeller proprietary



System Design Challenges

- Systems possess natural hierarchy
  - System
    - component assembly
      - component
        - material
          - microstructure - …

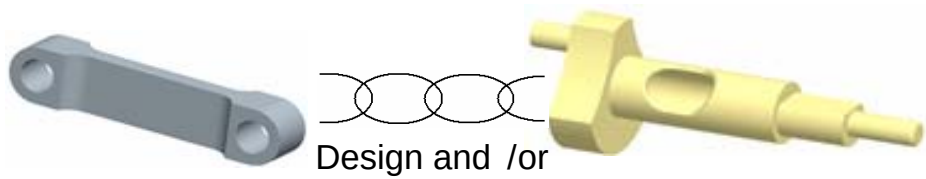


Jones et al., 2005

System Design Challenges

- Systems are expensive to analyze and therefore to synthesize
- Analysis requires exchange of subsystems' design
  - Geometry/Material data exchange compromises proprietary information
- Interactions are through geometrical and behavioral constraints

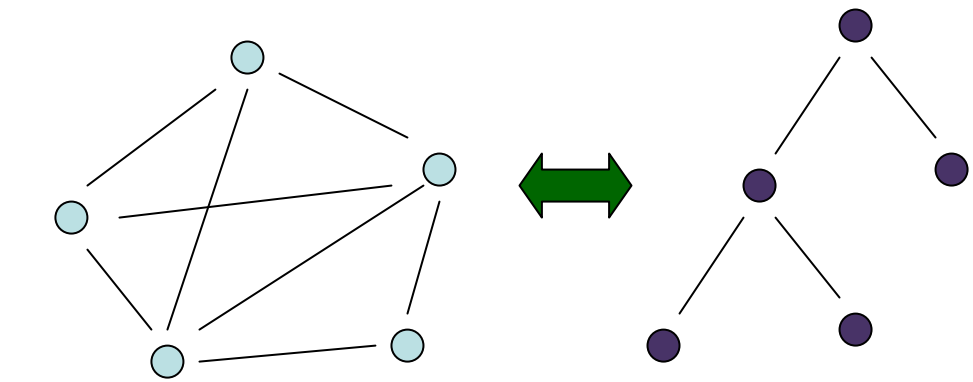
	FRIGATE*	COLLINS CLASS SUBMARINE	AIR WARFARE DESTROYER*
Mean displacement (tonnes)	3,600	3,400	6,000
Number of systems	60	108	120
Number of suppliers	600	1,600	1,500
Crew size	163	43	180
Patrol duration (hours)	340	1,700	430
Number of parts to assemble	170,000	500,000	600,000
Number of hours to assemble	1,200,000	2,500,000	3,800,000
Construction time (months)	22	60	42
Price (A$million)	600	1,000	1,500



Goal

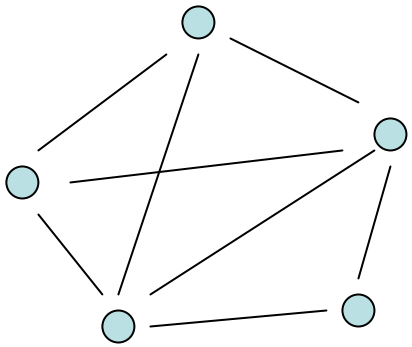
Isogeometric methodology for distributed as well as hierarchical synthesis of systems

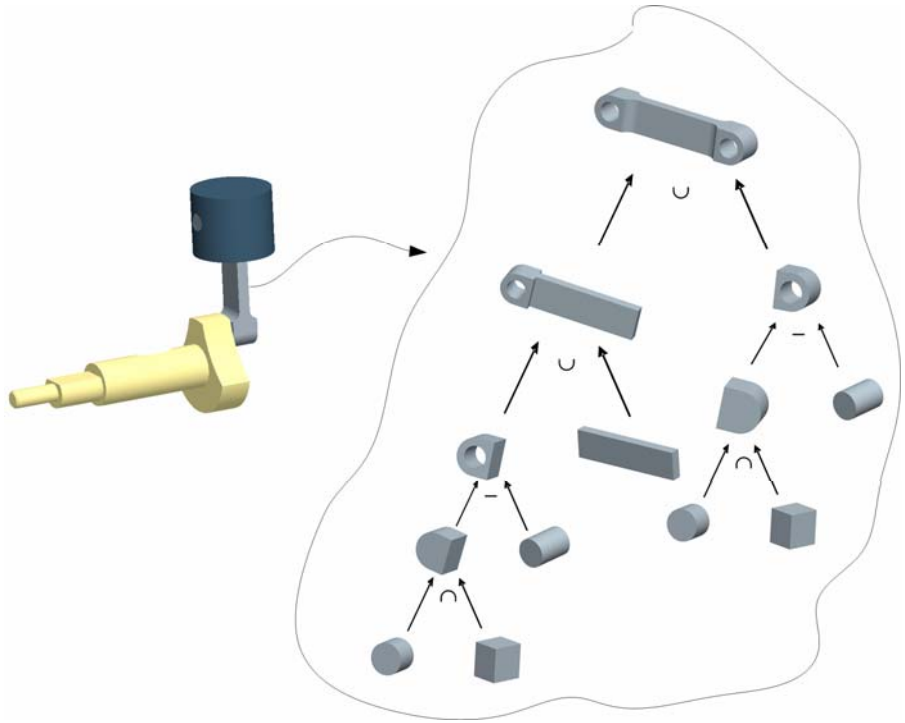
- Hierarchy: System – component assembly – component – material – microstructure - …
- Hierarchical procedure for sensitivities to subsystem design variations
- Hierarchical procedure for estimating uncertainties to subsystem variations



Compositions for Component Assemblies

Assembly and Component Design State

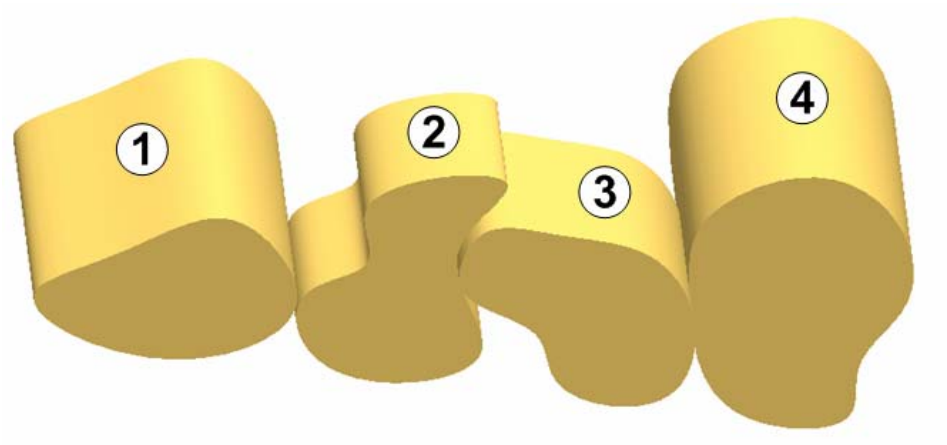




Assembly state $(\phi^i, \phi^j \ldots)$

Design state $\phi^i \in D^i$

System Design State



$$\Omega = \Omega^1 \cup \Omega^2 \cup \ldots \cup \Omega^{n_c}$$
$$\Omega^i \cap \Omega^j = \varnothing \qquad i \neq j$$

$$\mathcal{D} = \left\{ \Phi = \left(\boldsymbol{\phi}^1, \ldots, \boldsymbol{\phi}^{n_c} \right), \boldsymbol{\phi}^i \in \mathcal{D}^i(\Omega^i),\ i = 1, 2, \ldots, n_c \right\}$$

$$\mathcal{U} = \left\{ \mathbf{U} = \left(\mathbf{u}^1, \ldots, \mathbf{u}^{n_c} \right), \mathbf{u}^i \in \mathcal{U}^i(\Omega^i),\ i = 1, 2, \ldots, n_c \right\}$$

Interaction Constraints

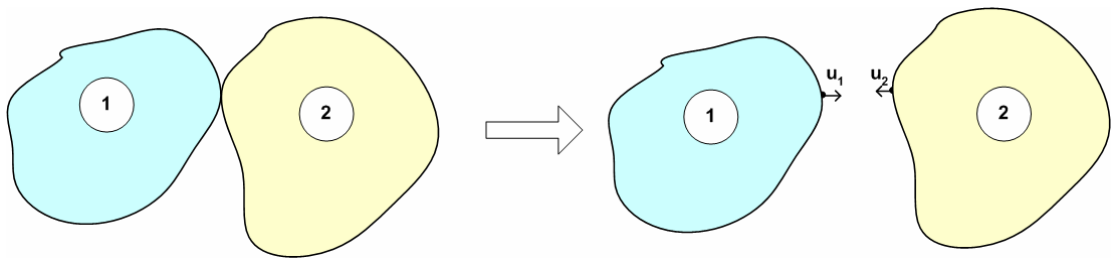
- Design constraint over boundary:

$$D(\Phi) : \mathcal{D}(\Gamma) \rightarrow \mathrm{R}$$

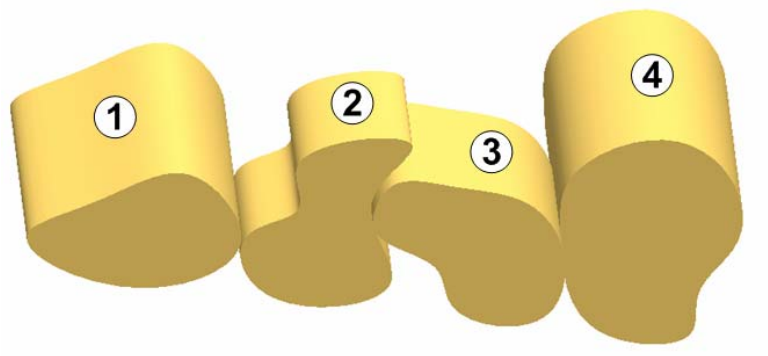
Hole Dia < 2 cm

- Behavior constraint over boundary:

$$B(\mathbf{U}) : \mathcal{U}(\Gamma) \rightarrow \mathrm{R}$$



Component Assembly Solution System



$$\begin{bmatrix} K_1 & & & & G_1^T \\ & K_2 & & & G_2^T \\ & & K_3 & & G_3^T \\ & & & K_4 & G_4^T \\ G_1 & G_2 & G_3 & G_4 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ -d \end{Bmatrix}$$

Two-Level Solution Procedure

Inspired by FETI, *Farhat and Roux, 1991*

Global

$$\begin{bmatrix} \mathbf{A} & \mathbf{-H} \\ \mathbf{-H^T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

Local

$$\mathbf{u}^1 = \mathbf{K}^{1^+}\left(\mathbf{q}^1 + \mathbf{G}^{1^T}\boldsymbol{\lambda}\right) + \mathbf{R}^1\boldsymbol{\alpha}$$

$$\mathbf{u}^2 = \mathbf{K}^{2^+}\left(\mathbf{q}^2 - \mathbf{G}^{T}_{2}\boldsymbol{\lambda}\right) + \mathbf{R}^2\boldsymbol{\alpha}$$

Sub-systems require only λ and α

$$\mathbf{A} = \mathbf{A}^1 + \mathbf{A}^2 = \left(\mathbf{G}^1\mathbf{K}^{1^+}\mathbf{G}^{1^T} + \mathbf{G}^2\mathbf{K}^{2^+}\mathbf{G}^{2^T}\right)$$

$$\mathbf{c} = \mathbf{c}^1 + \mathbf{c}^2 = \mathbf{G}^1\mathbf{K}^{1^{-1}}\mathbf{q}^1 + \mathbf{G}^2\mathbf{K}^{2^+}\mathbf{q}^2$$

$$\mathbf{d} = -\mathbf{R}^{1^T}\mathbf{q}^1 - \mathbf{R}^{2^T}\mathbf{q}^2$$

$$\mathbf{H} = \mathbf{G}^1\mathbf{R}^1 + \mathbf{G}^2\mathbf{R}^2$$

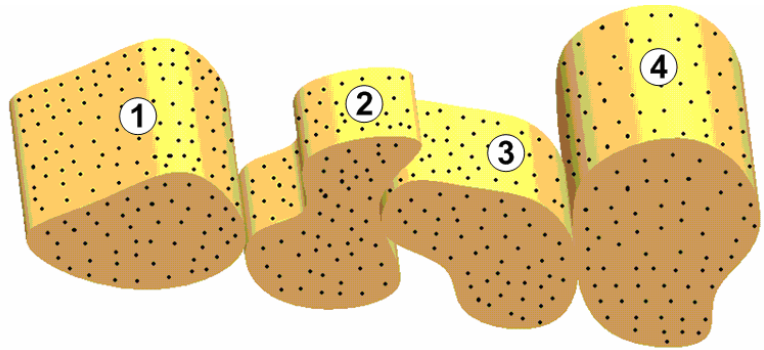
Sensitivity to Component Design Change

Distributed Sensitivity: Direct Differentiation Method

$$g(\phi, u)$$

$$\frac{dg}{d\phi} = \frac{\partial g}{\partial \phi} + \frac{\partial g}{\partial u}^{T} \left(\frac{\partial u}{\partial \phi} \right)$$

$$\mathbf{u}^{i} = \mathbf{K}^{i^{+}} \left(\mathbf{q}^{i} - \mathbf{G}^{i^{\mathrm{T}}} \boldsymbol{\lambda} \right) + \mathbf{R}^{i} \boldsymbol{\alpha}$$



$$\frac{\partial \mathbf{u}^{i}}{\partial \phi} = \mathbf{K}^{i^{+}} \frac{\partial \mathbf{q}^{i}}{\partial \phi} - \mathbf{K}^{i^{+}} \mathbf{G}^{i^{\mathrm{T}}} \frac{\partial \boldsymbol{\lambda}}{\partial \phi} - \mathbf{K}^{i^{+}} \frac{\partial \mathbf{G}^{i^{\mathrm{T}}}}{\partial \phi} \boldsymbol{\lambda} - \mathbf{K}^{i^{+}} \frac{\partial \mathbf{K}^{l}}{\partial \phi} \mathbf{u}^{i} + \mathbf{R}^{i} \frac{\partial \boldsymbol{\alpha}}{\partial \phi}$$

Sub-System i

$$\begin{bmatrix} \mathbf{A} & -\mathbf{H} \\ -\mathbf{H}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{\partial \boldsymbol{\lambda}}{\partial \phi} \\ \frac{\partial \boldsymbol{\alpha}}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \mathbf{e} \\ \mathbf{p} \end{bmatrix}$$

Local	Global	Global
$\mathbf{K}^{i^{+}} \left(\frac{\partial \mathbf{q}^{i}}{\partial \phi} - \frac{\partial \mathbf{K}^{i}}{\partial \phi} \mathbf{u}^{i} - \frac{\partial \mathbf{G}^{i}}{\partial \phi} \boldsymbol{\lambda} \right)$	$\mathbf{G}^{i} \mathbf{K}^{i^{+}} \left(\frac{\partial \mathbf{q}^{i}}{\partial \phi} - \frac{\partial \mathbf{K}^{i}}{\partial \phi} \mathbf{u}^{i} - \frac{\partial \mathbf{G}^{i}}{\partial \phi} \boldsymbol{\lambda} \right) + \mathbf{u}^{i} \frac{\partial \mathbf{G}^{i}}{\partial \phi}$	$\mathbf{e}^{i}$
	$\mathbf{R}^{i^{\mathrm{T}}} \left(\frac{\partial \mathbf{q}^{i}}{\partial \phi} - \boldsymbol{\lambda} \frac{\partial \mathbf{G}^{i^{\mathrm{T}}}}{\partial \phi} \right) + \frac{\partial \mathbf{R}^{i^{\mathrm{T}}}}{\partial \phi} \left(\mathbf{q}^{i} - \mathbf{G}^{i^{\mathrm{T}}} \boldsymbol{\lambda} \right)$	$\mathbf{p}^{i}$

Data Encapsulation

Uncertainty Evaluation

- Evaluation using First order Second Moment method

$$\delta \mathbf{g}^i = \frac{\partial g^i}{\partial \phi} \delta \phi^i + \frac{\partial g^i}{\partial u^i}^T \left(\frac{\partial u^i}{\partial \phi} \right) \delta \phi^i + \frac{\partial g^i}{\partial \sigma^i}^T \left(\frac{\partial \sigma^i}{\partial \phi} \right) \delta \phi^i$$

$$\delta \mathbf{g}^1 = \frac{\partial g^i}{\partial \phi} \delta \phi^i + \frac{\partial g^i}{\partial u^i}^T \delta u^i + \frac{\partial g^i}{\partial \sigma^i}^T \delta \sigma^i$$

Need $\delta \phi^2$ to evaluate δu^1 -- suppliers may be unwilling to share $\delta \phi^2$!

Global Uncertainty Evaluation

$$\begin{bmatrix} \mathbf{A} & \mathbf{-H}^{2} \\ \mathbf{-H}^{2^{\mathbf{T}}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{\partial \boldsymbol{\lambda}}{\partial \phi^{i}} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{e} \\ \mathbf{p}^{2} \end{bmatrix}$$

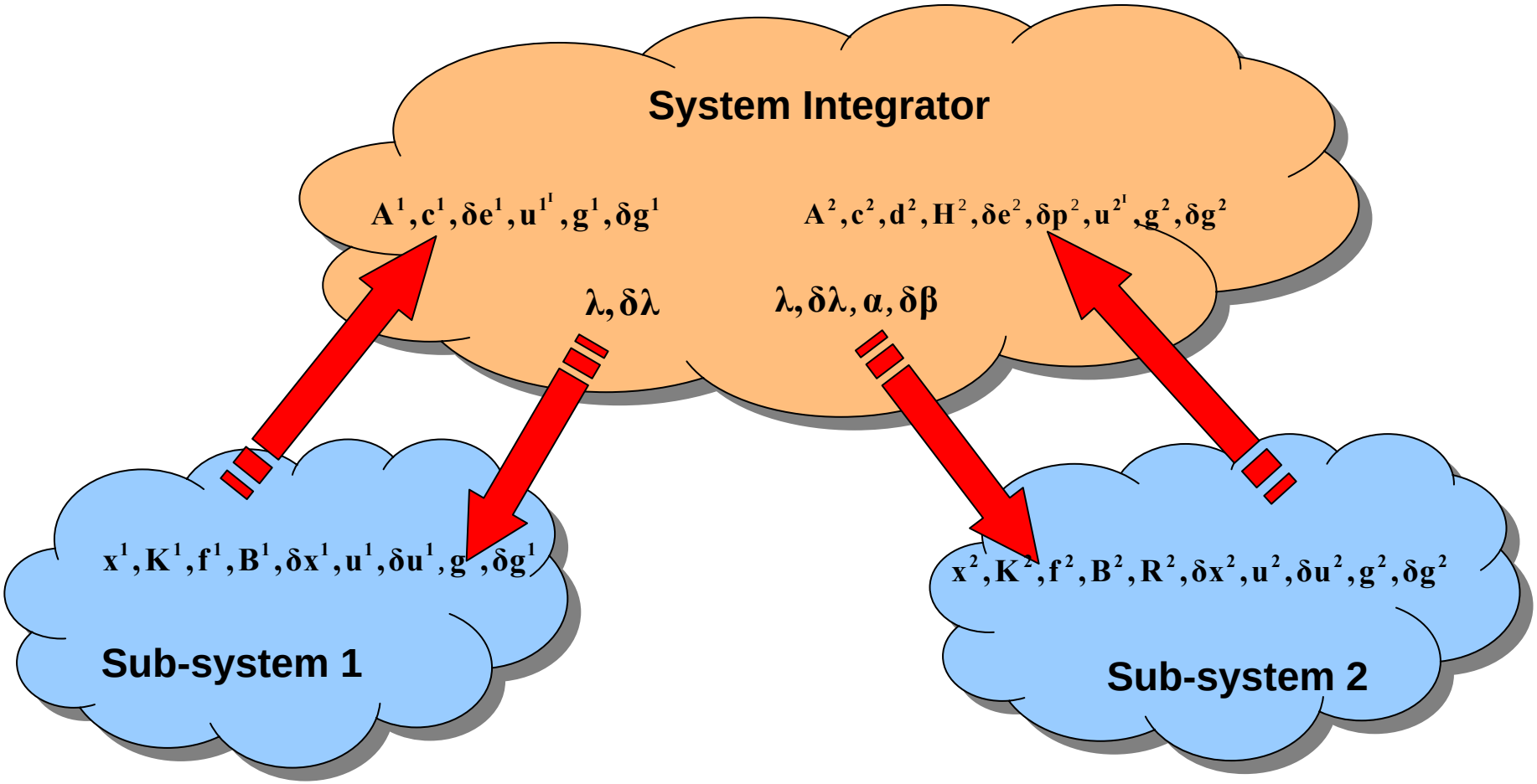
$$\begin{bmatrix} \mathbf{A} & \mathbf{-H}^{2} \\ \mathbf{-H}^{2^{\mathbf{T}}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\lambda} \\ \delta\boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \delta\mathbf{e} \\ \delta\mathbf{p}^{2} \end{bmatrix}$$

$$\delta\mathbf{e} = -\delta\mathbf{e}^{1} + \delta\mathbf{e}^{2} = -\mathbf{B}^{1}\mathbf{K}^{1^{-1}} \left(\frac{\partial \mathbf{q}^{1}}{\partial \phi^{i}} - \frac{\partial \mathbf{K}^{1}}{\partial \phi^{i}} \mathbf{u}^{1} \right) \delta x + \mathbf{B}^{2}\mathbf{K}^{2^{+}} \left(\frac{\partial \mathbf{q}^{2}}{\partial \phi^{i}} - \frac{\partial \mathbf{K}^{2}}{\partial \phi^{i}} \mathbf{u}^{2} \right) \delta\phi^{i}$$

$$\delta\mathbf{p}^{2} = \left[\mathbf{-R}^{2^{\mathbf{T}}} \frac{\partial \mathbf{q}^{2}}{\partial \phi^{i}} \right] \delta\phi^{i}$$

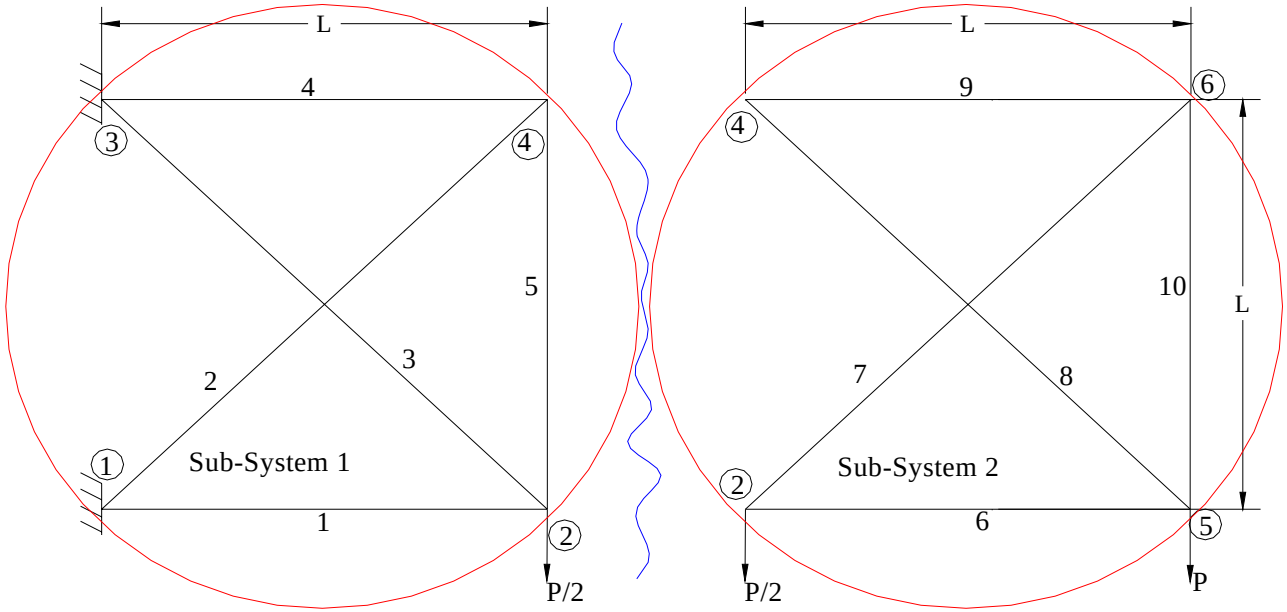
$$\delta\boldsymbol{\beta} = \boldsymbol{\beta}\,\delta\phi^{i}$$

Exchange of Information



Demonstration

Results: Ten Bar Truss



- Linking function: Displacement $u_{5y} \leq u_{all}$
- Behavior : Displacement
  - Partitioning of the behavior
  - Local evaluation of system function
  - Local evaluation of sensitivity values

Solution

- P = 1000 KN
- A = 0.005 m^2
- E = 210 GPa
- L = 1
- $\lambda = \{-1\times10^{6}, -9.01\times10^{5}, 1\times10^{6}, -5.99\times10^{5}\}$
- $\alpha = \{-9.5\times10^{-3}, 2.39\times10^{-3}, -2.57\times10^{-3}\}$

$$\mathbf{u}^1 = \{-1.95\times10^{-3}, -4.77\times10^{-3}, 1.86\times10^{-3}, -4.43\times10^{-3}\}$$

$$\mathbf{u}^2 = \begin{Bmatrix} -1.95\times10^{-3}, -4.77\times10^{-3}, 1.86\times10^{-3}, -4.43\times10^{-3}, \\ -2.52\times10^{-3}, -1.04\times10^{-2}, 2.24\times10^{-3}, -1\times10^{-2} \end{Bmatrix}$$

Local evaluation of displacement!

Worst Case Design

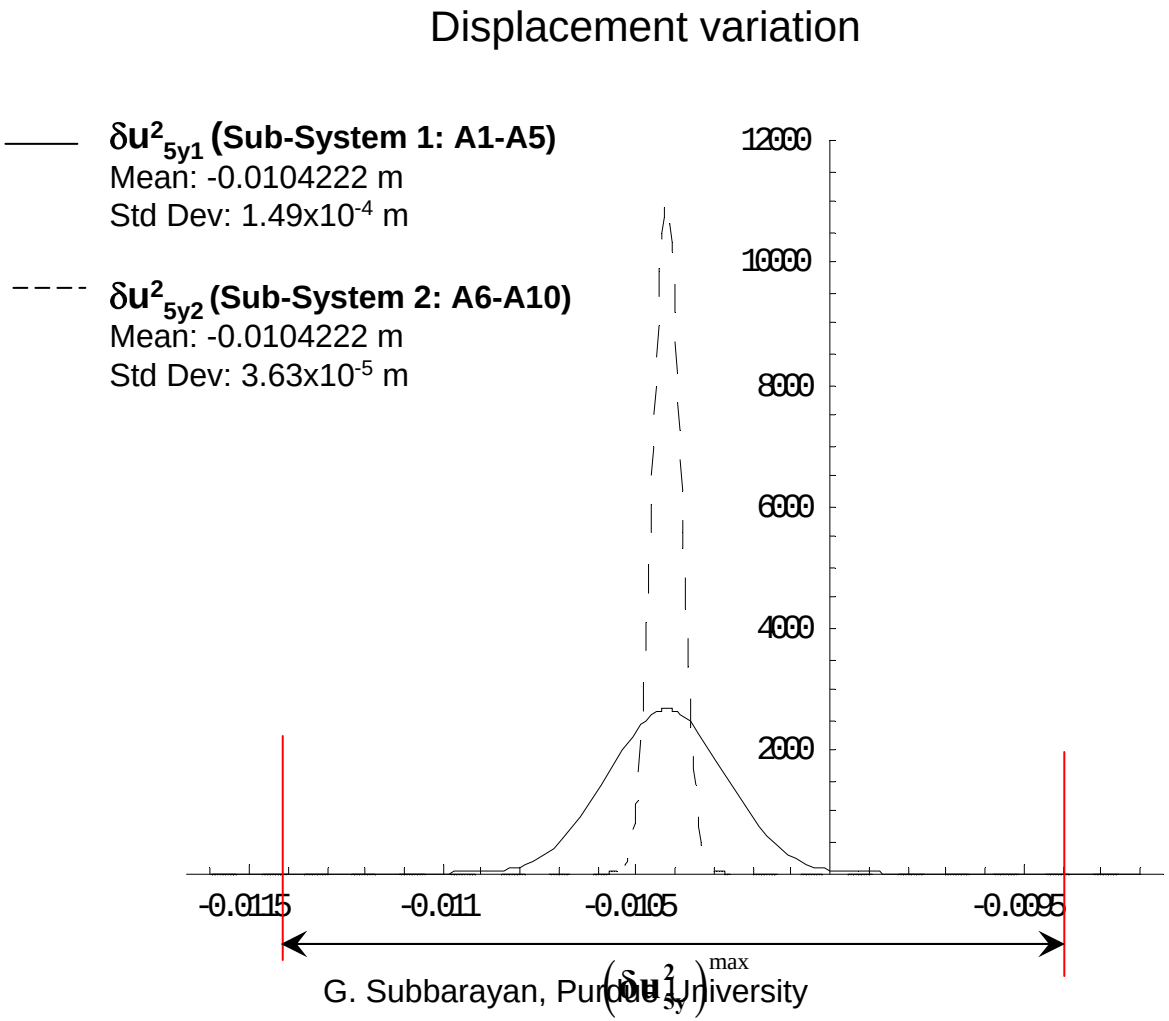
- Largest $\delta\lambda$ for both the sub-systems
  - Supply $\boldsymbol{\delta\lambda}_8$ to both sub-systems
  - Determine max change in sub-system functions

$$\left(\boldsymbol{\delta u}_{2y}^{1}\right)^{\max} = \{5.33\times10^{-5}\} \qquad \left(\boldsymbol{\delta u}_{5y}^{2}\right)^{\max} = \{9.86\times10^{-5}\}$$

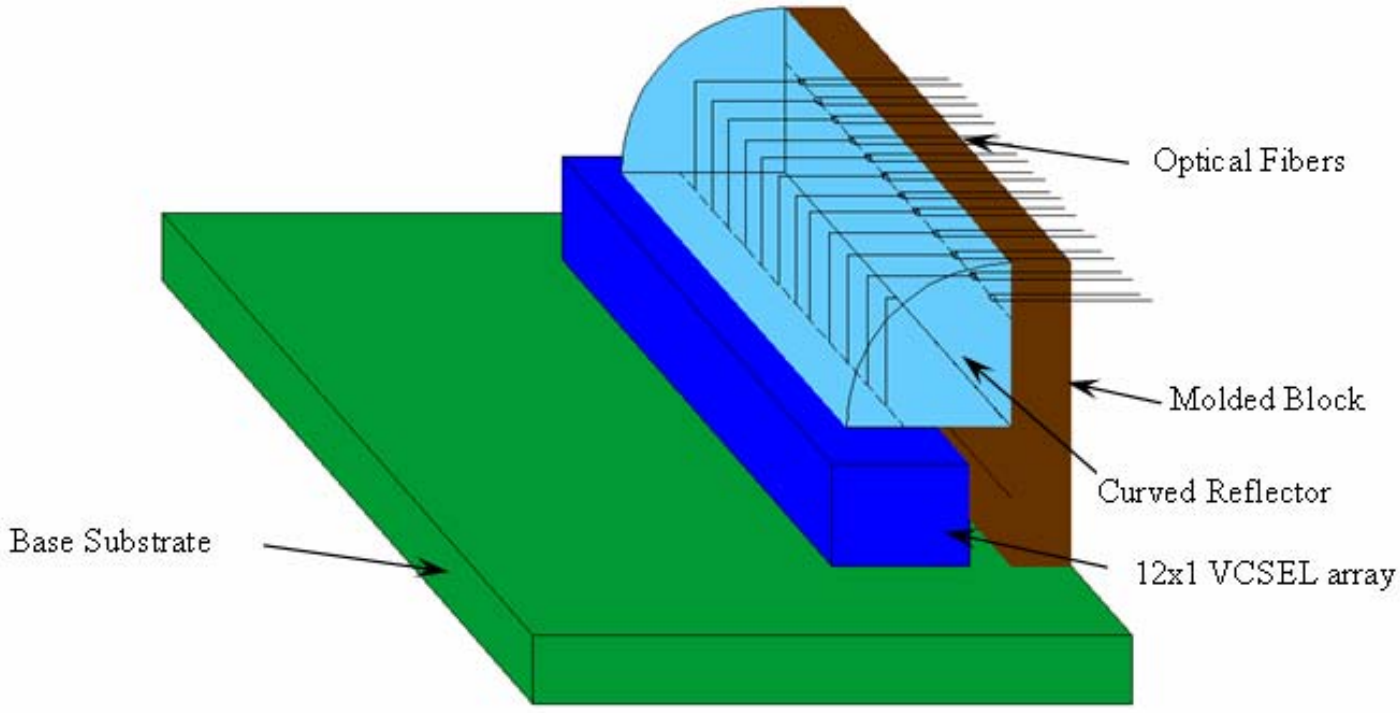
$$\left(\boldsymbol{\delta u}_{y}\right)^{\max} = \{9.86\times10^{-5}\}$$

Uncertainty Analysis

- **Displacement variation due to variation in areas**



Photonics Problem

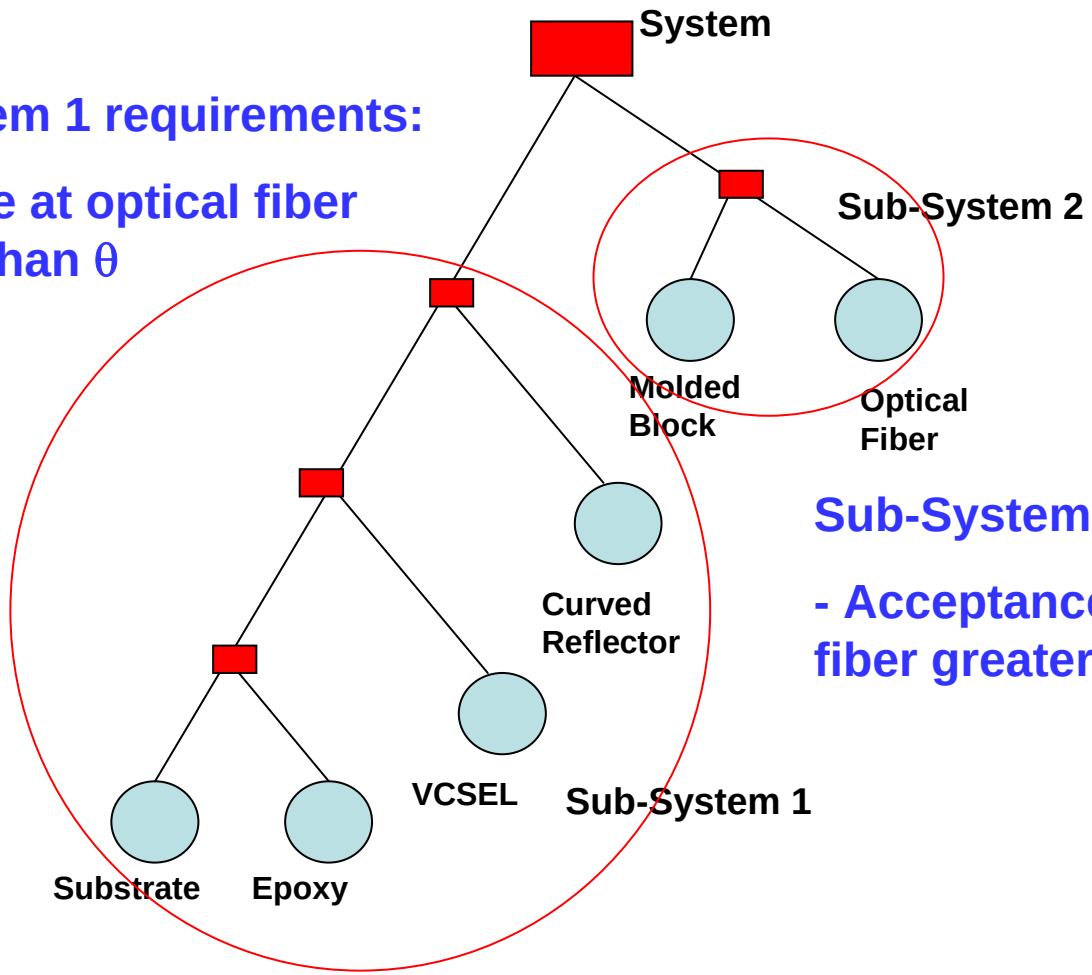


- Behavior: Divergence angle at optical fiber end
  - Coupling efficiency is maximum when the divergence angle is less than the acceptance angle

Distributed Design

Sub-System 1 requirements:

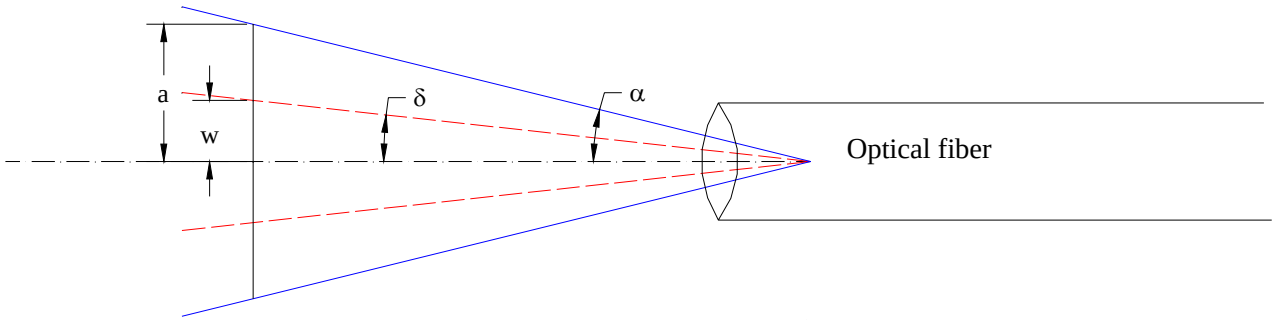
- Div angle at optical fiber end less than θ



Sub-System 2 requirements:

- Acceptance angle of optical fiber greater than θ

Divergence Angle at Optical Fiber End



$$f \equiv \left(1 - e^{-2(a/w)^2}\right)$$

- Coupling efficiency

$$w = x\,\mathrm{Tan}(\delta) \qquad a = x(\mathrm{Tan}(\alpha))$$

Coupling efficiency of first and second sub-system

$$f \equiv \left(1 - e^{-2\left(\frac{\mathrm{Tan}(\alpha)}{\mathrm{Tan}(\delta)}\right)^2}\right)$$

Solution

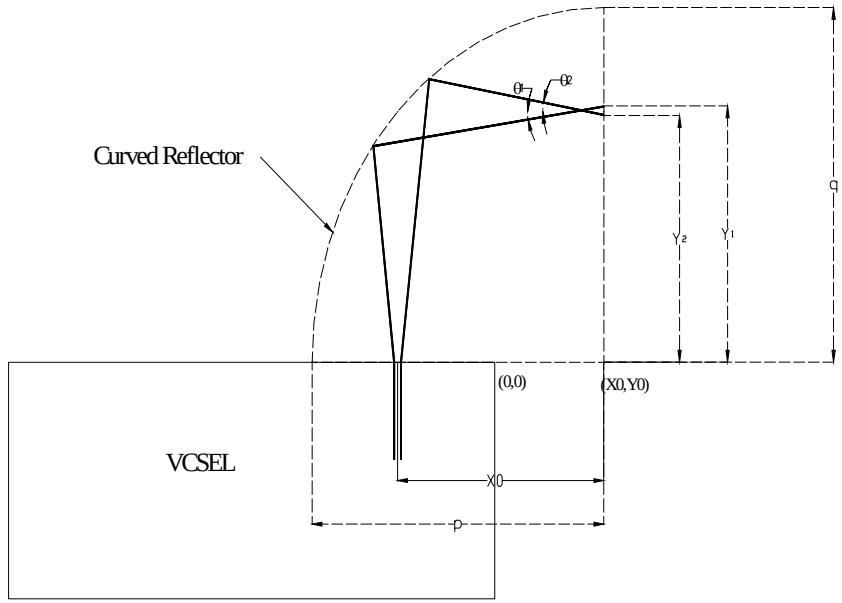
Optimal design of first sub-system

Method	p (μm)	q (μm)	x0 (μm)	Coupling Efficiency %
Deterministic Optimal Design	600	270	545	100

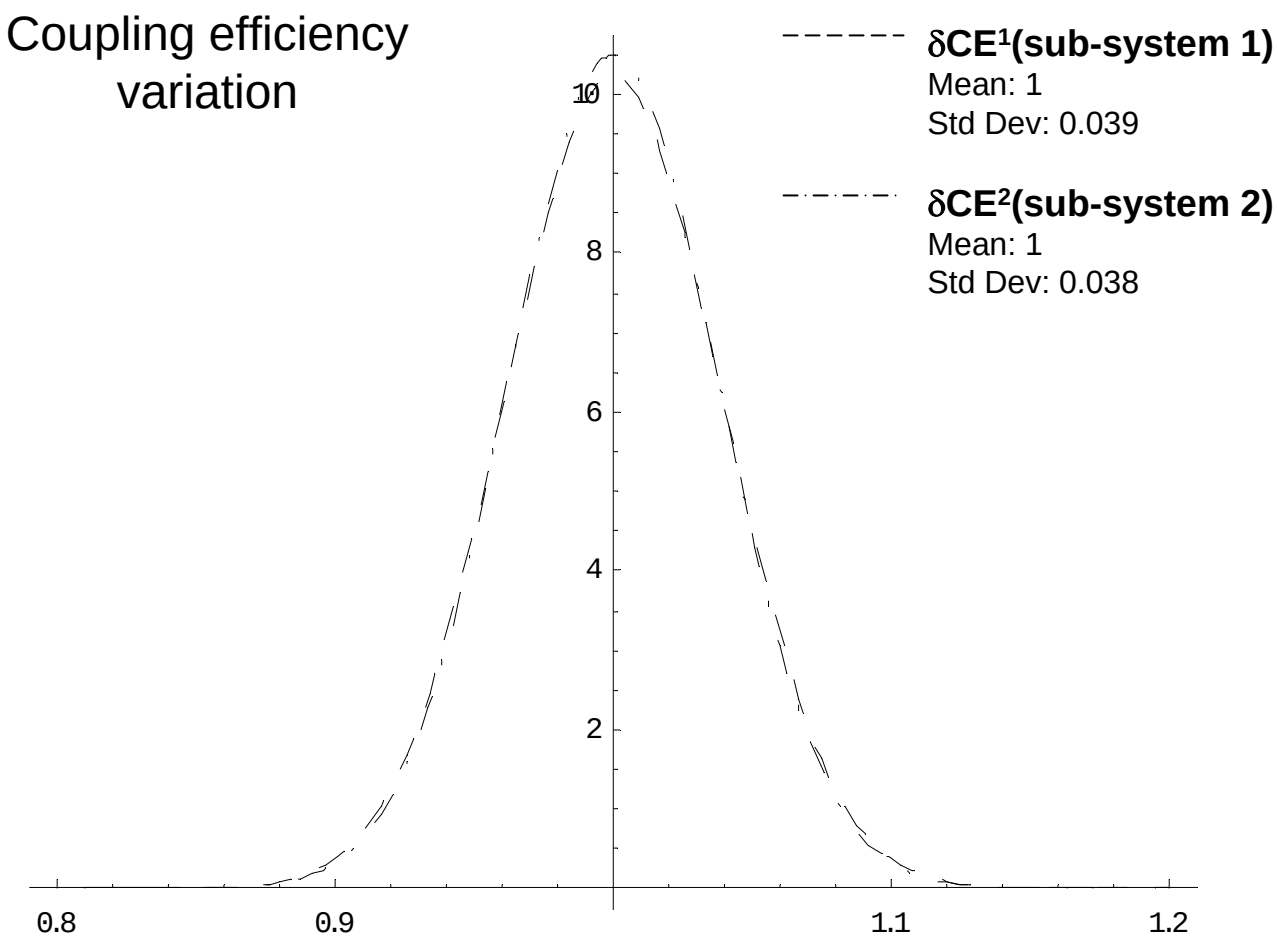
Sensitivity analysis

df^1/dp	df^1/dq	df^1/dx_0	df^1/ddx
9118	-2901	-8525	-52631

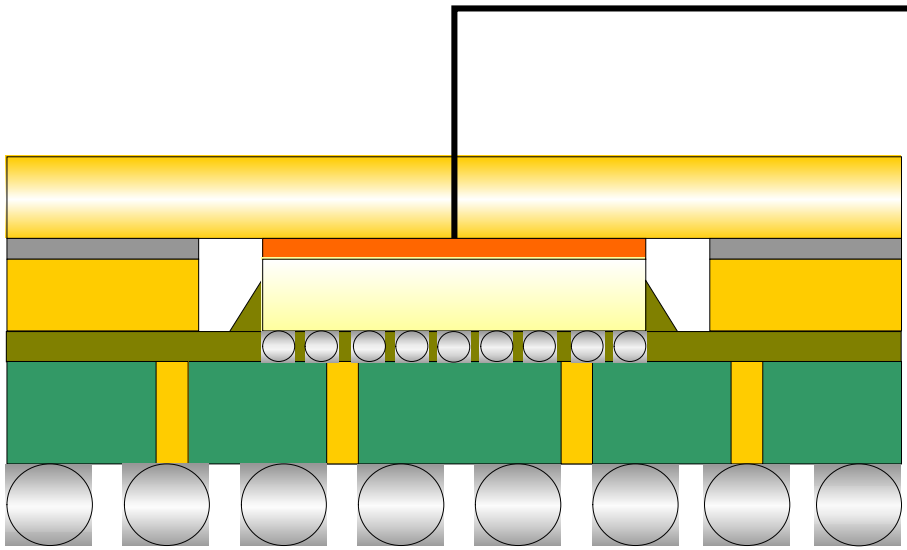
df^2/dNA	df^2/dw_0	df^2/ddx
2.23	6579	-52631

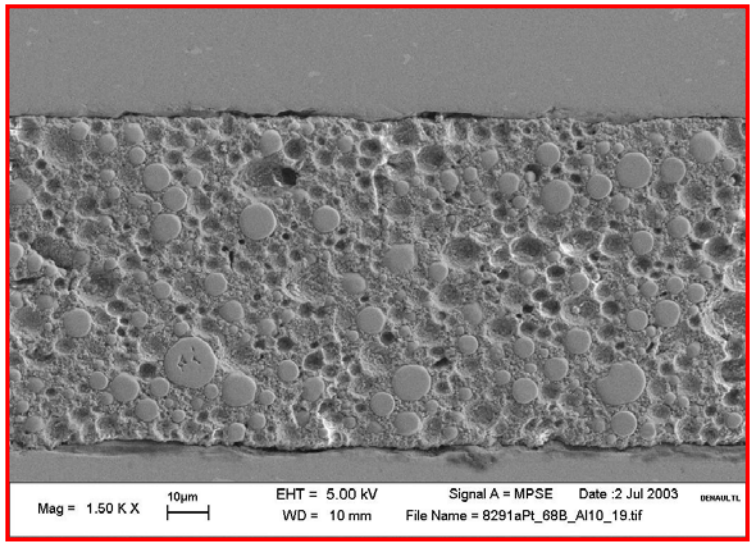


Uncertainty analysis



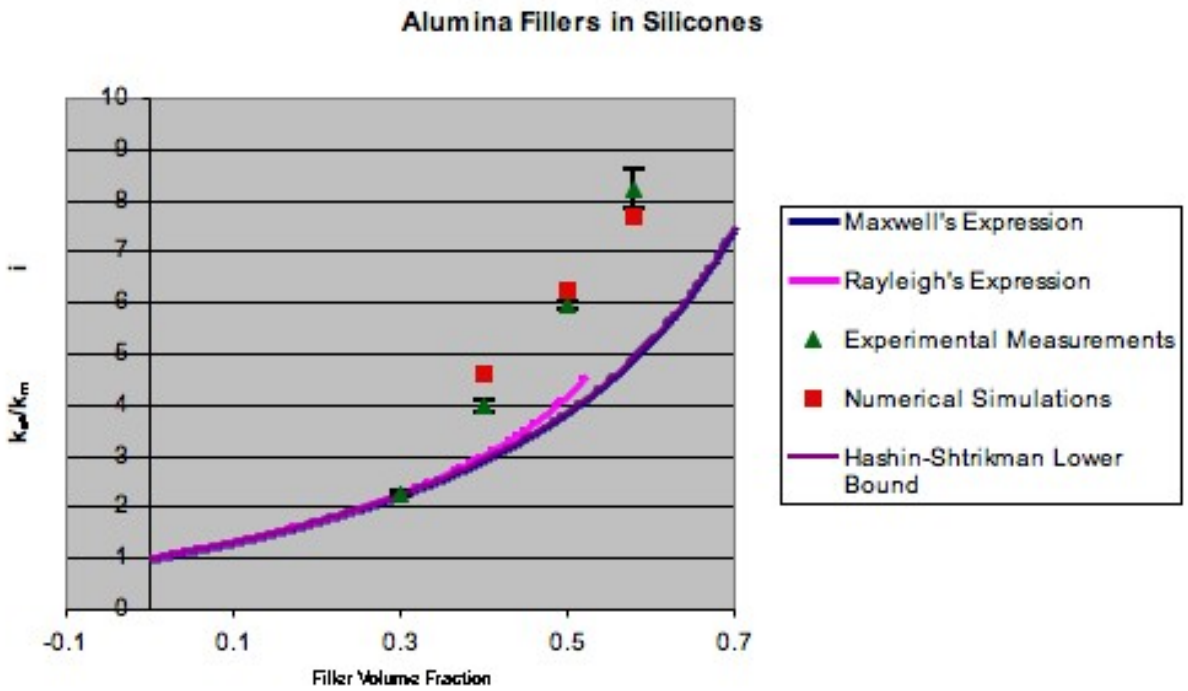
Particulate Interface Materials

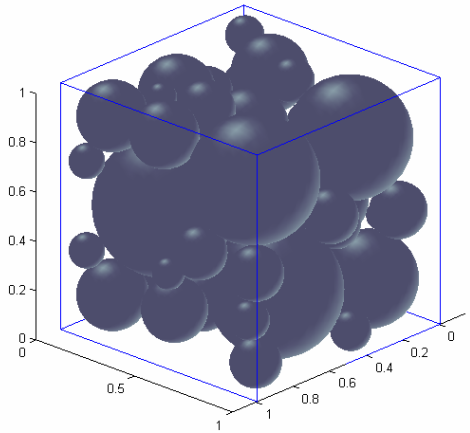




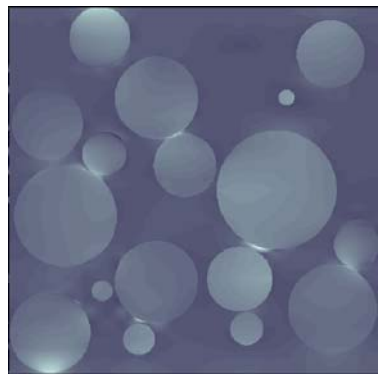
3D Simulations

- Silicone matrix (k=0.2 W/mK), with Alumina fillers (k=25 W/mK)





Random 3D microstructure



Heat flux field on y=0.5

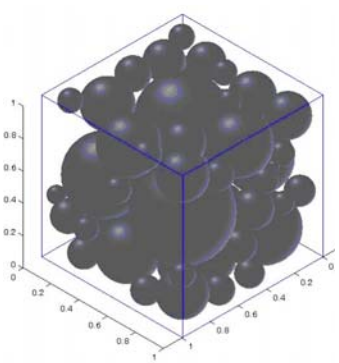
Particulate Composite Solution System

- Material dominated union between the matrix and particles
- Temperature continuity constraints on the shared boundaries

$$\begin{bmatrix} K_1 & & & & G_1^T \\ & K_2 & & & G_2^T \\ & & K_3 & & G_3^T \\ & & & K_4 & G_4^T \\ G_1 & G_2 & G_3 & G_4 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ -d \end{Bmatrix}$$

Distributed Sensitivity Results

Number of particles	Volume fraction (%)	$\frac{d\mathbf{k}_{\text{eff}}}{dk_M}$ Parallel solver (PS)	$\frac{d\mathbf{k}_{\text{eff}}}{dk_M}$ Finite differences (FD)	$\frac{d\mathbf{k}_{\text{eff}}}{dk_M}$ Maxwell's model	% Error
1	6.54	1.2	1.2	1.2	0
2	12.78	2.134	2.133	1.416	0.05
30	60.31	7.467	7.170	5.048	4.14
40	60.12	7.620	7.948	5.018	-4.13
50	60.15	7.814	7.912	5.023	-1.24
60	58.03	7.049	7.319	4.706	-3.69



Maxwell's Model:

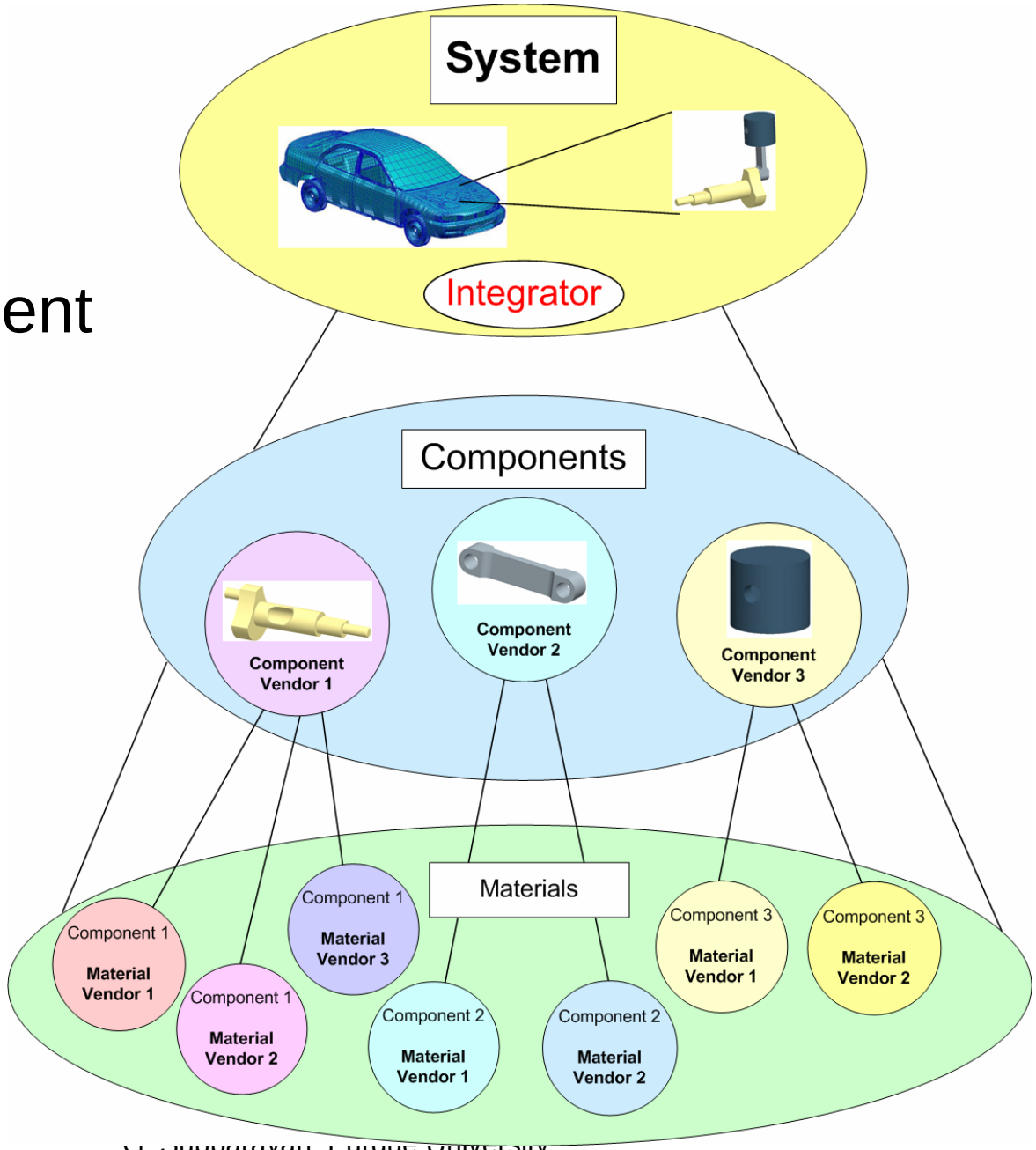
$$\frac{k_{eff}}{k_m} = \frac{k_P + 2k_M + 2f\left[k_P - k_M\right]}{k_P + 2k_M - f\left[k_P - k_M\right]}$$

Summary

- Developed a formal procedure for Distributed Design of Engineering Systems without geometry/material data exchange
- Developed techniques to propagate design sensitivity across the system using the Adjoint and Direct Differentiation Method
- Evaluated uncertainty at the system level using the uncertainty in the design variable at the component level
- Demonstrated the developed formalism using a ten bar truss example, photonics system and particulate composite material system

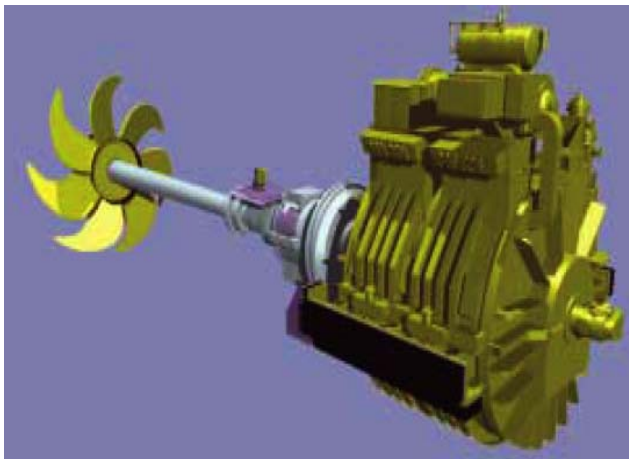
Thanks!

Lengthscale Hierarchy Inherent in Systems

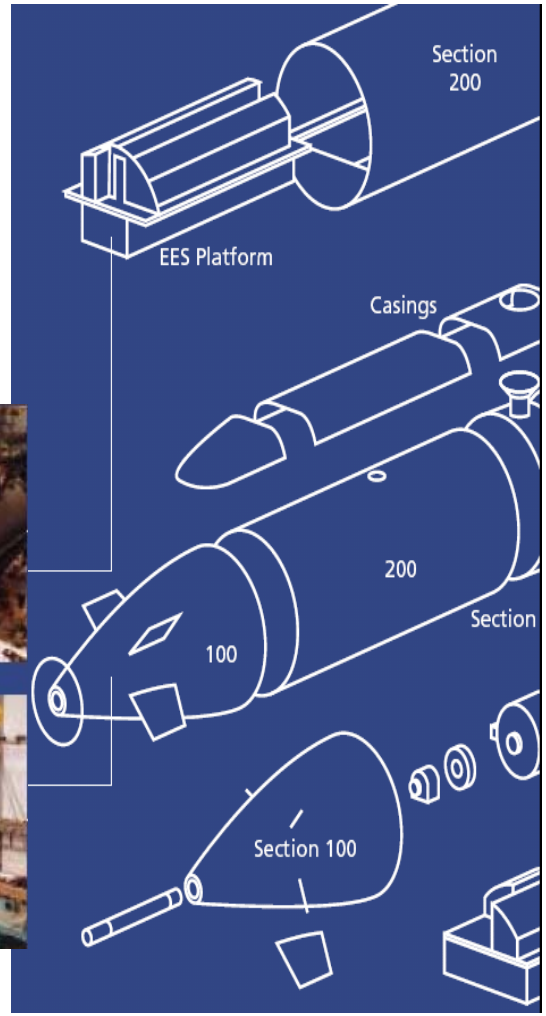


Propulsion System

Propulsion System



System Components



Hierarchical Partition of Unity Constructions for Components

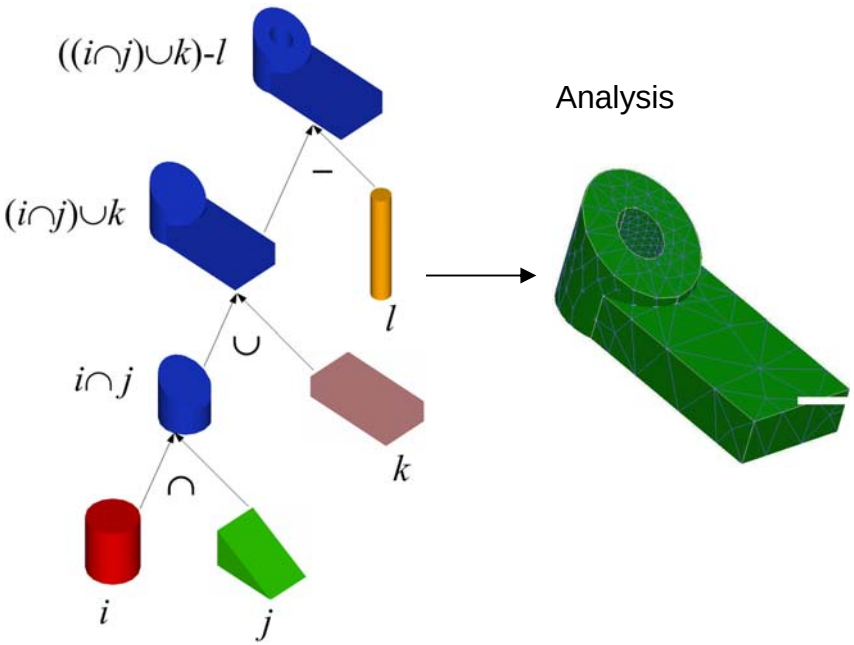
Rayasam M, Srinivasan V, Subbarayan G. CAD Inspired Hierarchical Partition of Unity Constructions for NURBS-based, Meshless Design, Analysis and Optimization. *International Journal for Numerical Methods in Engineering*. (In Press).

CAD Inspired Compositions

(Natekar 2002 and Zhang 2004)

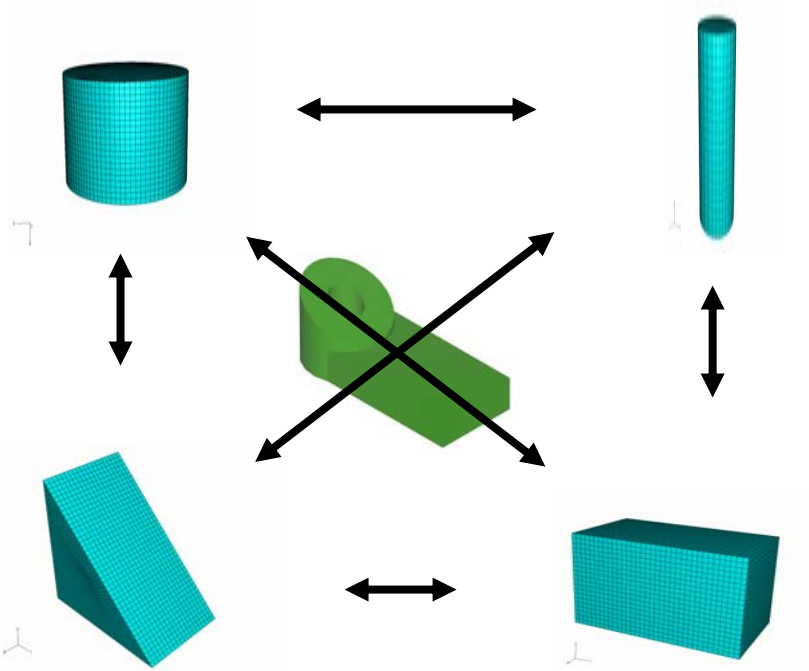
- Conventional Analysis Schemes

Geometry construction



Analyze the final component

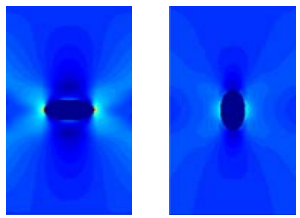
- Each primitive is discretized and interactions analyzed



Discretize the primitives

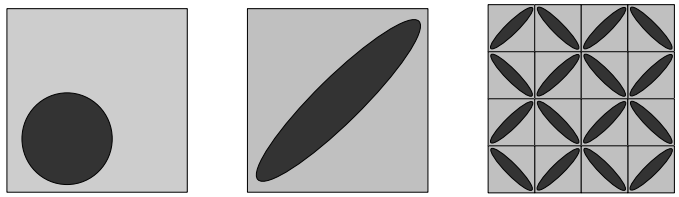
Isogeometric Analysis and Synthesis

Shape optimization



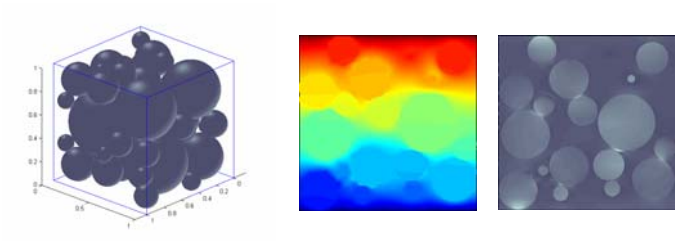
Natekar et al., CAD, 2004

Microstructure design



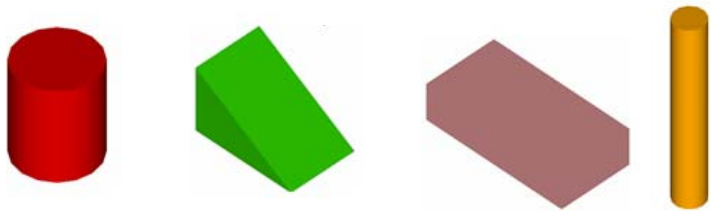
Zhang, 2004

Particulate composite



Zhang and Subbarayan, AES, 2006

Design State in a Domain



Design state $D_i = (G_i, M_i)$

Geometry

Material

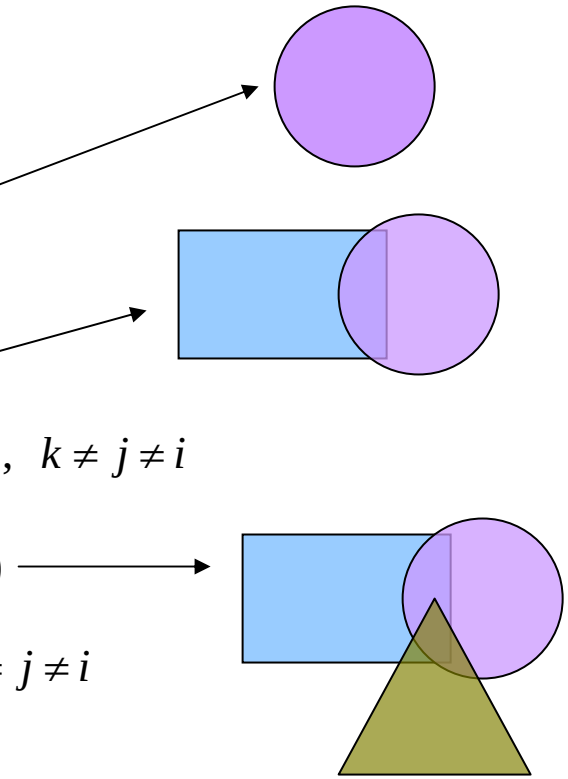
$$G_i \subset H^1(\Omega_i \cap \Omega) \qquad M_i \subset H^0(\Omega_i \cap \Omega)$$

Global Design Space: $\mathcal{D} = \left\{ \mathbf{d} = \left(d^1, ..., d^{n_p} \right), d^i \in \mathcal{D}^i(\Omega^i),\ i = 1, 2, ..., n_p \right\}$

Hierarchical Composition

Global field: $\tilde{f}(\mathbf{x}) = \sum_i w^i(\mathbf{x}) f^i(\mathbf{x}), \quad f^i \in \mathcal{F}^i$

$$
w^i(\mathbf{x}) = \begin{cases}
0 \ \text{if} \ \mathbf{x} \notin \Omega^i \\
1 \ \text{in} \ \Omega^i - \bigcup_j (\Omega^i \cap \Omega^j) \quad j = 1,2...,n_p, \quad j \neq i \\
w^{ij}(\mathbf{x}) \ \text{in} \ \left(\Omega^i \cap \Omega^j\right) - \bigcup_k (\Omega^i \cap \Omega^j \cap \Omega^k) \quad j,k = 1,2...,n_p, \quad k \neq j \neq i \\
w^{(ij)k}(\mathbf{x}) w^{ij}(\mathbf{x}) \ \text{in} \ \left(\Omega^i \cap \Omega^j\right) \cap \Omega^k - \bigcup_l (\Omega^i \cap \Omega^j \cap \Omega^k \cap \Omega^l) \\
\qquad\qquad j,k,l = 1,2...,n_p, \quad l \neq k \neq j \neq i \\
\vdots
\end{cases}
$$



Partition of Unity: $\sum_i w^i(\mathbf{x}) = 1 \quad \text{in} \ \Omega, \quad 0 \leq w^i \leq 1$

Partition of Unity Constructions

If

$$\left\| f - f^i \right\|_{L^2(\Omega^i)} \leq \varepsilon_1(i)$$

$$\left\| \nabla\left(f - f^i \right) \right\|_{L^2(\Omega^i)} \leq \varepsilon_2(i)$$

$$\left\| w^i \right\|_{L^\infty(R^d)} \leq C_\infty$$

$$\left\| \nabla w^i \right\|_{L^\infty(R^d)} \leq \frac{C_G}{\operatorname{diam} \Omega^i}$$

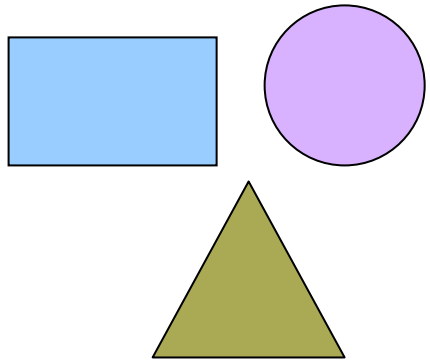
then

Melenk and Babuska 1996

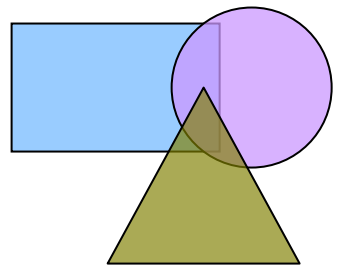
$$\left\| f - \tilde{f} \right\|_{L^2(\Omega)} = \left\| \sum_i w^i \left(f - f^i \right) \right\|_{L^2(\Omega)} \leq \sqrt{n_p} C_\infty \left(\sum_i \varepsilon_1^2(i) \right)^{\frac{1}{2}}$$

$$\left\| \nabla\left(f - \tilde{f} \right) \right\|_{L^2(\Omega)} = \left\| \nabla\left[\sum_i w^i \left(f - f^i \right) \right] \right\|_{L^2(\Omega)}$$

$$\leq \sqrt{2n_p} \left\{ \sum_i \left[\left(\frac{C_G}{\operatorname{diam} \Omega^i} \right)^2 \varepsilon_1^2(i) + C_\infty^2 \varepsilon_2^2(i) \right] \right\}^{\frac{1}{2}}$$



Local approximation spaces



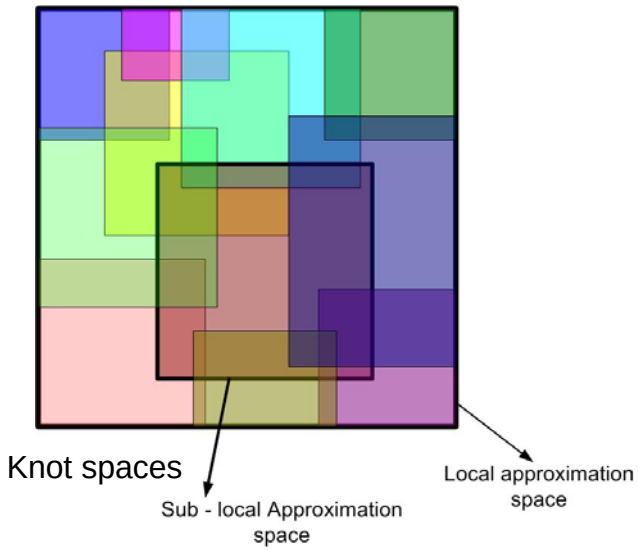
Global approximation space

Recursive Partitions of Unity using NURBS

$$\sum_I N_I = 1$$

$$\left\| N_I \right\|_{L^\infty(R^d)} \leq C_\infty^N$$

$$\left\| \nabla N_I \right\|_{L^\infty(R^d)} \leq \frac{C_G^N}{\mathrm{diam}\Omega^{N_I}}$$

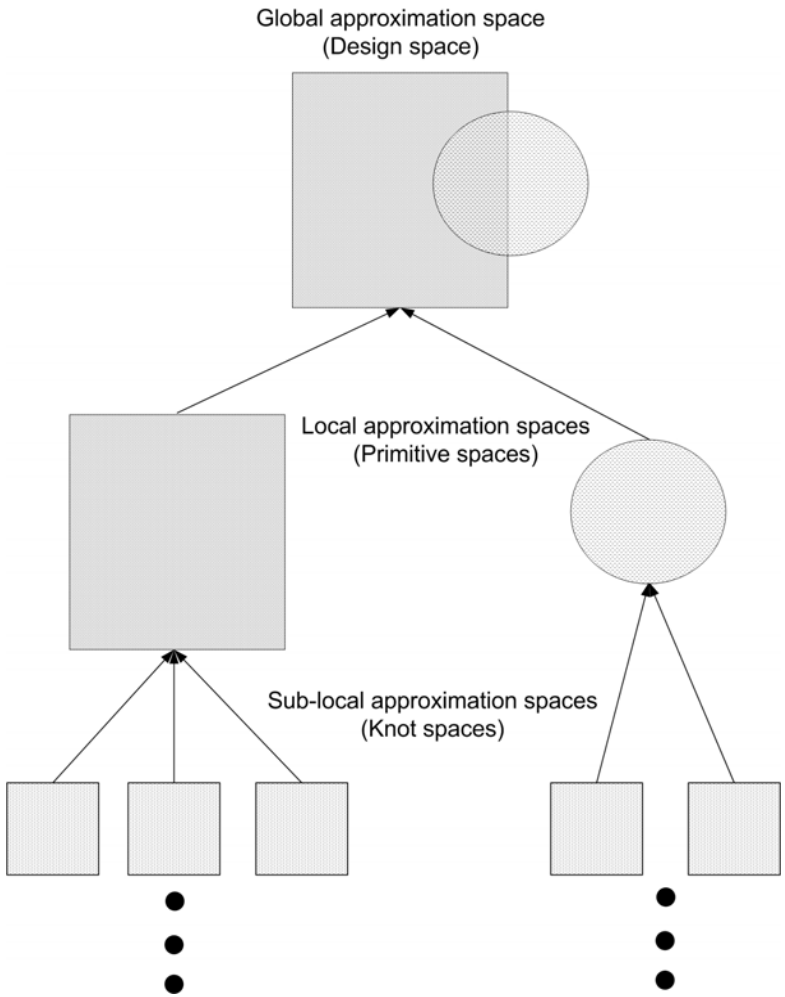


Field over each primitive

$$\tilde{f}(\xi,\eta,\zeta) = \sum_I N_I(\xi,\eta,\zeta)\, f_I$$

f_I is defined over the knot space

Summary: Hierarchical PU Framework



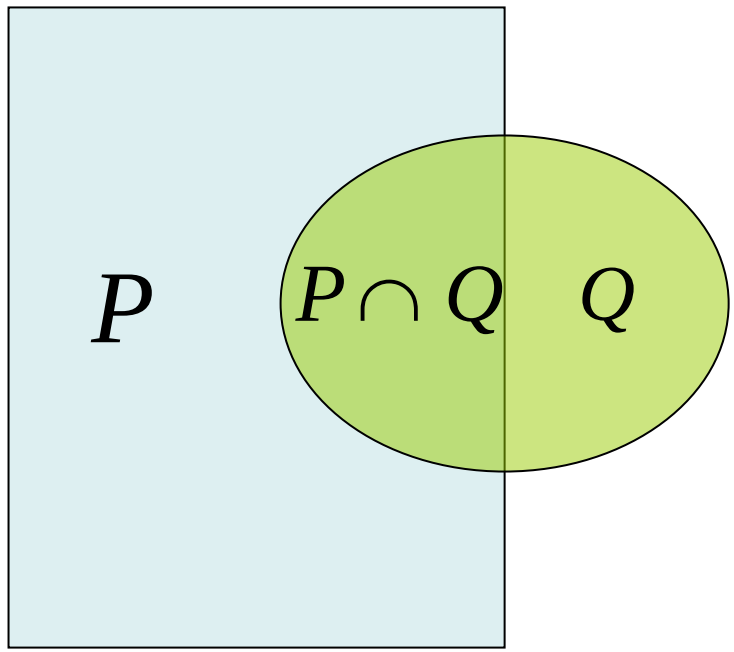
Multiscale: Hierarchical construction of approximation spaces

Multiresolution: Construction of NURBS spaces

Component-Level Solution System

- Discretized form similar to FEA

$$\begin{bmatrix} [K^P] & [K^{P\cap Q}] \\ [K^{P\cap Q}] & [K^Q] \end{bmatrix} \begin{bmatrix} u^P \\ u^Q \end{bmatrix} = \begin{bmatrix} q^P \\ q^Q \end{bmatrix}$$



- Amenable to iterative solution without matrix assembly

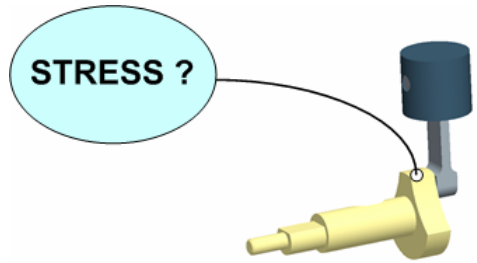
$$\left[K^P\right]\left[u^P\right] = \left[q^P - [K^{P\cap Q}]\left\{u^Q\right\}\right]$$

$$\left[K^Q\right]\left[u^Q\right] = \left[q^Q - [K^{P\cap Q}]\left\{u^P\right\}\right]$$

System Functions

- System Function:

$$g(\Phi, \mathbf{U}(\Phi)) : \mathcal{D}(\Omega) \times \mathcal{U}(\Omega) \rightarrow \mathrm{R}$$



$$g(\Phi, \mathbf{U}(\Phi), \boldsymbol{\sigma}(\mathbf{U}(\Phi))) = \int_{\Omega} \psi(\Phi, \mathbf{U}(\Phi), \boldsymbol{\sigma}(\mathbf{U}(\Phi)))d\Omega + \int_{\Gamma_u} \zeta(\Phi, \mathbf{U_0}, \mathbf{T})d\Gamma + \int_{\Gamma_T} \xi(\Phi, \mathbf{U}(\Phi), \mathbf{T_0})d\Gamma$$

$$\int_{\Omega} \psi(\Phi, \mathbf{U}(\Phi), \boldsymbol{\sigma}(\mathbf{U}(\Phi)))d\Omega = \sum_i \int_{\Omega} \psi^i(\Phi^i, \mathbf{U}^i(\Phi), \boldsymbol{\sigma}^i(\mathbf{U}(\Phi)))d\Omega$$

$$\int_{\Gamma_u} \zeta(\Phi, \mathbf{U_0}, \mathbf{T})d\Gamma = \sum_{\Gamma \cap \Gamma_u^i} \int_{\Gamma_u^i} \zeta^i(\Phi^i, \mathbf{U_0^i}, \mathbf{T}^i)d\Gamma$$

$$\int_{\Gamma_T} \xi(\Phi, \mathbf{U}(\Phi), \mathbf{T_0})d\Gamma = \sum_{\Gamma \cap \Gamma_T^i} \int_{\Gamma_T} \xi^i(\Phi^i, \mathbf{U}^i(\Phi), \mathbf{T_0^i})d\Gamma$$

Augmented System Function

$$\mathcal{L} = \sum_i \left\{ \begin{array}{l} \int_{\Omega^i} \psi^i d\Omega + \int_{\Gamma_u^i} \zeta^i d\Gamma + \int_{\Gamma_T^i} \xi^i d\Gamma \\ + \sum_{j \neq i} \int_{\Gamma^{\overline{\Omega^i \supset \subset \Omega^j}}} \nu^{ij} d^{ij} d\Gamma + \sum_{j \neq i} \int_{\Gamma^{\overline{\Omega^i \supset \subset \Omega^j}}} \gamma^{ij} b^{ij} d\Gamma \end{array} \right\}$$

Interfacial Design Constraint

Lagrange Multipliers

Interfacial Behavior Constraint

Adjoint Sensitivity

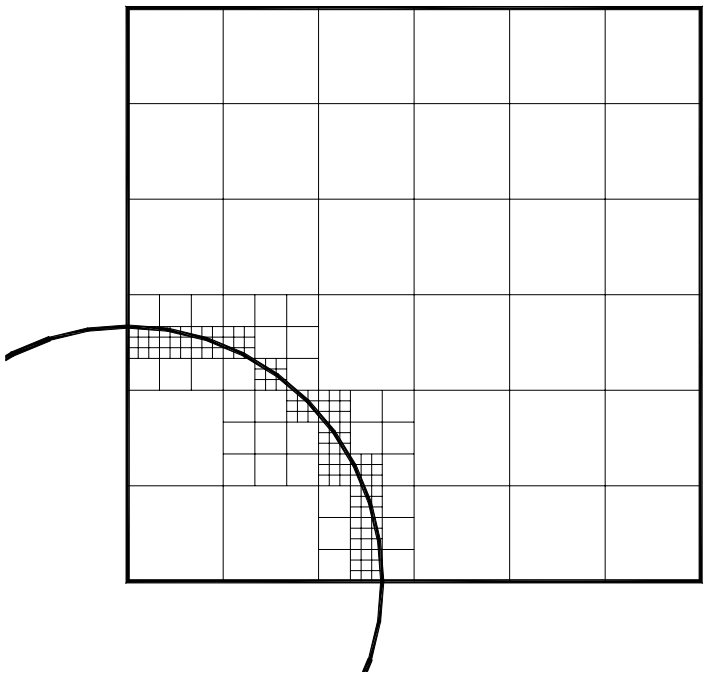
$$L(\boldsymbol{\phi}^{\mathbf{i}}, \mathbf{u}^{\mathbf{i}}, \lambda^{i}, \upsilon^{ij}, \mathbf{t}^{\mathbf{ij}}) = \sum_{i} \left\{ g^{i}(\boldsymbol{\phi}^{\mathbf{i}}, \mathbf{u}^{\mathbf{i}}) - \lambda^{i}(K^{i}\mathbf{u}^{\mathbf{i}} + \sum_{j \neq i} G^{ij^{T}}\mathbf{t}^{\mathbf{ij}} - f^{i}) - \sum_{j \neq i} \upsilon^{ij}(G^{ij}\mathbf{u}^{\mathbf{i}} - G^{ji}\mathbf{u}^{\mathbf{j}}) \right\}$$

$$\dot{L} = \sum_{i} \left\{ \begin{array}{l} \left(g^{i}_{,\phi^{i}} - \lambda^{i}\left[K^{i}_{,\phi^{i}}\mathbf{u}^{\mathbf{i}} + \sum_{j \neq i}\left(G^{ij^{T}}_{,\phi^{i}}\mathbf{t}^{\mathbf{ij}} \right) - f^{i}_{,\phi^{i}} \right] - \sum_{j \neq i} \upsilon^{ij}\left(G^{ij}_{,\phi^{i}}\mathbf{u}^{\mathbf{i}} - G^{ji}_{,\phi^{i}}\mathbf{u}^{\mathbf{j}} \right) \right) \\ + \left(g^{i}_{,u^{i}} - \lambda^{i}K^{i} - \sum_{j \neq i} \upsilon^{ij}G^{ij} \right)\dot{\mathbf{u}}^{\mathbf{i}} - \left(K^{i}\mathbf{u}^{\mathbf{i}} + \sum_{j \neq i} G^{ij^{T}}\mathbf{t}^{\mathbf{ij}} - f^{i} \right)\dot{\lambda}^{i} - \\ \sum_{j \neq i}\left(\left(\lambda^{i}G^{ij^{T}} \right)\dot{\mathbf{t}}^{\mathbf{ij}} \right) - \sum_{j \neq i} (G^{ij}\mathbf{u}^{\mathbf{i}} - G^{ji}\mathbf{u}^{\mathbf{j}})\dot{\upsilon}^{ij} \end{array} \right\}$$

$$\frac{dg^{i}}{d\boldsymbol{\phi}^{\mathbf{i}}} = \frac{\partial g^{i}}{\partial \boldsymbol{\phi}^{\mathbf{i}}} - \lambda^{i}\left[\frac{\partial K^{i}}{\partial \boldsymbol{\phi}^{\mathbf{i}}}\mathbf{u}^{\mathbf{i}} + \sum_{j \neq i}\left(\frac{\partial G^{ij^{T}}}{\partial \boldsymbol{\phi}^{\mathbf{i}}}\mathbf{t}^{\mathbf{ij}} \right) - \frac{\partial f^{i}}{\partial \boldsymbol{\phi}^{\mathbf{i}}} \right] - \sum_{j \neq i} \upsilon^{ij}\left(\frac{\partial G^{ij}}{\partial \boldsymbol{\phi}^{\mathbf{i}}}\mathbf{u}^{\mathbf{i}} - \frac{\partial G^{ji}}{\partial \boldsymbol{\phi}^{\mathbf{i}}}\mathbf{u}^{\mathbf{j}} \right)$$

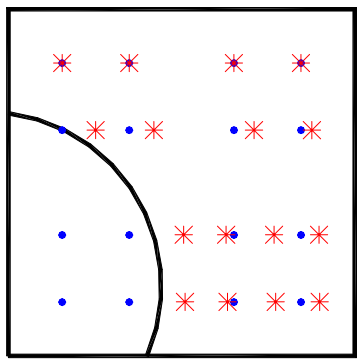
Challenges to Accurate analysis

- Adaptive Quadrature



Adaptive quadrature scheme

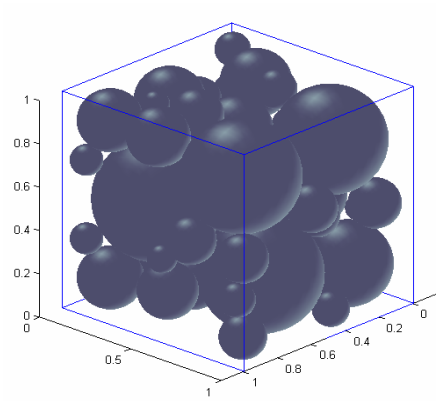
$$\alpha = \frac{A}{A_0}$$



Quadrature nodes adjustment

Sensitivity – Maxwell's model

- Maxwell's model
  - Valid for dilute fractions



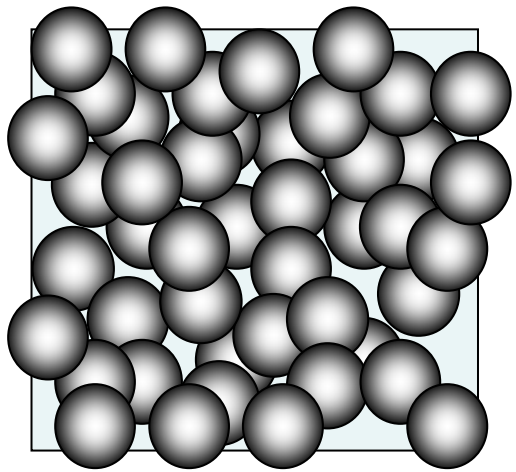
$$\frac{k_{eff}}{k_m} = \frac{k_P + 2k_M + 2f\left[k_P - k_M\right]}{k_P + 2k_M - f\left[k_P - k_M\right]}$$

$$\frac{dk_{eff}}{dk_P} = \frac{k_M\left(k_P + 2k_M - f\left[k_P - k_M\right]\right)(1+2f) - \left(k_P + 2k_M + 2f\left[k_P - k_M\right]\right)(1-f)}{\left(k_P + 2k_M - f\left[k_P - k_M\right]\right)^2}$$

$$\frac{dk_{eff}}{dk_M} = \frac{\left(k_P + 2k_M - f\left[k_P - k_M\right]\right)\left(k_P + 4k_M + 2f\left[k_P - 2k_M\right]\right) - k_M\left(k_P + 2k_M + 2f\left[k_P - k_M\right]\right)(2+f)}{\left(k_P + 2k_M - f\left[k_P - k_M\right]\right)^2}$$

Sensitivity to Matrix/Particle Conductivity

$$\frac{dk_{eff}}{dk_i} = \frac{\partial k_{eff}}{\partial k_i} + \frac{\partial k_{eff}}{\partial T}^T \left(\frac{\partial T}{\partial \phi} \right)$$



$$\mathbf{k_{eff}} = \frac{\int_{\Omega_{Total}} k \nabla T d\Omega}{\int_{\Omega_{Total}} \nabla T d\Omega} = \frac{\sum_{i=1}^{n} k_P \int_{\Omega_{P_i}} [B]\{T\} d\Omega + k_M \int_{\Omega_M} [B]\{T\} d\Omega}{\sum_{i=1}^{n} \int_{\Omega_{P_i}} [B]\{T\} d\Omega + \int_{\Omega_M} [B]\{T\} d\Omega}$$

Sensitivity to Conductivity Changes

$$T^i = \mathbf{K}^{i^+}\left(\mathbf{q}^i - \mathbf{G}^{i^{\mathrm{T}}}\boldsymbol{\lambda}\right) + \mathbf{R}^i\boldsymbol{\alpha}$$

$$\frac{\partial T^i}{\partial \boldsymbol{\phi}} = \mathbf{K}^{i^+}\frac{\partial \mathbf{q}^i}{\partial \boldsymbol{\phi}} - \mathbf{K}^{i^+}\mathbf{G}^{i^{\mathbf{T}}}\frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\phi}} - \mathbf{K}^{i^+}\frac{\partial \mathbf{G}^{i^{\mathbf{T}}}}{\partial \boldsymbol{\phi}}\boldsymbol{\lambda} - \mathbf{K}^{i^+}\frac{\partial \mathbf{K}^i}{\partial \boldsymbol{\phi}}T^i + \mathbf{R}^i\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\phi}}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{-H} \\ \mathbf{-H}^{\mathbf{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\phi}} \\ \dfrac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \mathbf{e} \\ \mathbf{p} \end{bmatrix}$$

Sensitivity to Matrix/Particle Conductivity

$$\frac{\partial \mathbf{k}_{\mathbf{eff}}}{\partial \{T\}} = (k_P - \mathbf{k}_{\mathbf{eff}}) \sum_{i=1}^{n} \int_{\Omega_{P_i}} [B] d\Omega + (k_M - \mathbf{k}_{\mathbf{eff}}) \int_{\Omega_M} [B] d\Omega$$

$$\frac{\partial \mathbf{k}_{\mathbf{eff}}}{\partial k_M} = \int_{\Omega_M} [B]\{T\} d\Omega \qquad \frac{\partial \mathbf{k}_{\mathbf{eff}}}{\partial k_P} = \sum_{i=1}^{n} \int_{\Omega_{P_i}} [B]\{T\} d\Omega$$